A10. The Product Rule. Worksheet 2

Note: $[f \times g]^/ = [f]^/ \times g + [g]^/ \times f$

Use the product rule to differentiate the functions below.

A10.1 $y = x^3e^{5x}$ • $[f \times g]^/ = [f]^/ \times g + [g]^/ \times f$ • $[x^3]^/ \times [e^{5x}] + [e^{5x}]^/ \times x^3$ • $3x^2 \times e^{5x} + 5e^{5x} \times x^3$ • $3x^2e^{5x} + 5x^3e^{5x}$ • $e^{5x}x^2(3 + 5x)$	**A10.6** $y = (x^3 + 3x^2 - 2x)e^x$ • -- • -- • -- • -- • --
A10.2 $y = (2x + 5)e^{3x}$ • -- • -- • -- • --	**A10.7** $y = \sqrt{x}e^{5x}$ • -- • -- • -- • -- • --
A10.3 $y = (3x + 1)^4e^{2x}$ • -- • -- • -- • -- • --	**A10.8** $y = \sqrt{x - 1}e^{x+2}$ • -- • -- • -- • -- • --
A10.4 $y = xe^x$ • -- • -- • -- • -- • --	**A10.9** $y = \frac{e^{5x}}{x}$ • -- • -- • -- • -- • --
A10.5 $y = xe^{x^2}$ • -- • -- • -- • -- • --	**A9.10** $y = \frac{3x^2}{e^{2x}}$ • -- • -- • -- • -- • --

A10. The Product Rule. Worksheet 2. Answers

Note: $[f \times g]^{/} = [f]^{/} \times g + [g]^{/} \times f$

Use the product rule to differentiate the functions below.

A10.1 $y = x^3 e^{5x}$

- $[f \times g]^{/} = [f]^{/} \times g + [g]^{/} \times f$
- $[x^3]^{/} \times [e^{5x}] + [e^{5x}]^{/} \times x^3$
- $3x^2 \times e^{5x} + 5e^{5x} \times x^3$
- $3x^2 e^{5x} + 5x^3 e^{5x}$
- $e^{5x} x^2 (3 + 5x)$

A10.2 $y = (2x + 5)e^{3x}$

- $[f \times g]^{/} = [f]^{/} \times g + [g]^{/} \times f$
- $[2x + 5]^{/} \times [e^{3x}] + [e^{3x}]^{/} \times [2x + 5]$
- $2 \times e^{3x} + 3e^{3x} \times [2x + 5]$
- $2e^{3x} + 3(2x + 5)e^{3x}$
- $(6x + 17)e^{3x}$

A10.3 $y = (3x + 1)^4 e^{2x}$

- $[f \times g]^{/} = [f]^{/} \times g + [g]^{/} \times f$
- $[(3x + 1)^4]^{/} \times [e^{2x}] + [e^{2x}]^{/} \times [(3x + 1)^4]$
- $12(3x + 1)^3 \times e^{2x} + 2e^{2x} \times [(3x + 1)^4]$
- $12(3x + 1)^3 e^{2x} + 2(3x + 1)^4 e^{2x}$
- $(3x + 1)^3 e^{2x}[12 + 2(3x + 1)]$
- $(3x + 1)^3 e^{2x}[6x + 14]$

A10.4 $y = xe^x$

- $[f \times g]^{/} = [f]^{/} \times g + [g]^{/} \times f$
- $[x]^{/} \times [e^x] + [e^x]^{/} \times x$
- $1 \times e^x + e^x \times x$
- $e^x + xe^x$
- $(x + 1)e^x$

A10.5 $y = xe^{x^2}$

- $[f \times g]^{/} = [f]^{/} \times g + [g]^{/} \times f$
- $[x]^{/} \times [e^{x^2}] + [e^{x^2}]^{/} \times x$
- $1 \times e^{x^2} + 2x \times e^{x^2} \times x$
- $e^{x^2} + 2x^2 e^{x^2}$
- $(2x^2 + 1)e^{x^2}$

A10.6 $y = (x^3 + 3x^2 - 2x)e^x$

- $[f \times g]^{/} = [f]^{/} \times g + [g]^{/} \times f$
- $[x^3 + 3x^2 - 2x]^{/} e^x + [e^x]^{/} \times (x^3 + 3x^2 - 2x)$
- $(3x^2 + 6x - 2)e^x + e^x(x^3 + 3x^2 - 2x)$
- $(x^3 + 6x^2 + 4x - 2)e^x$

A10.7 $y = \sqrt{x} e^{5x} = x^{0.5} e^{5x}$

- $[f \times g]^{/} = [f]^{/} \times g + [g]^{/} \times f$
- $[x^{0.5}]^{/} \times [e^{5x}] + [e^{5x}]^{/} \times x^{0.5}$
- $0.5x^{-0.5} \times e^{5x} + 5e^{5x} \times x^{0.5}$
- $0.5x^{-0.5} e^{5x} + 5x^{0.5} e^{5x}$
- $e^{5x}(0.5x^{-0.5} + 5x^{0.5})$
- $e^{5x}(\frac{1}{2\sqrt{x}} + 5\sqrt{x})$

A10.8 $y = \sqrt{x - 1} e^{x+2} = (x - 1)^{0.5} e^{x+2}$

- $[f \times g]^{/} = [f]^{/} \times g + [g]^{/} \times f$
- $[(x - 1)^{0.5}]^{/} \times e^{x+2} + [e^{x+2}]^{/} \times (x - 1)^{0.5}$
- $0.5(x - 1)^{-0.5} e^{x+2} + e^{x+2}(x - 1)^{0.5}$
- $e^{x+2}[0.5(x - 1)^{-0.5} + (x - 1)^{0.5}]$
- $e^{x+2}(\frac{1}{2\sqrt{x-1}} + \sqrt{x - 1})$

A10.9 $y = \frac{e^{5x}}{x} = e^{5x} \times x^{-1}$

- $[f \times g]^{/} = [f]^{/} \times g + [g]^{/} \times f$
- $[e^{5x}]^{/} \times [x^{-1}] + [x^{-1}]^{/} \times e^{5x}$
- $5e^{5x} \times x^{-1} + (-1)x^{-2} \times e^{5x}$
- $5e^{5x} x^{-1} - x^{-2} e^{5x}$
- $e^{5x}(\frac{5}{x} - \frac{1}{x^2})$

A9.10 $y = \frac{3x^2}{e^{2x}} = 3x^2 \times e^{-2x}$

- $[f \times g]^{/} = [f]^{/} \times g + [g]^{/} \times f$
- $[3x^2]^{/} \times [e^{-2x}] + [e^{-2x}]^{/} \times 3x^2$
- $6x \times e^{-2x} - 2e^{-2x} \times 3x^2$
- $6xe^{-2x} - 6x^2 e^{-2x}$
- $6xe^{-2x}(1 - x) = \frac{6x(1-x)}{e^{2x}}$

M28. Finding the Equation of the Normal to the Curve

Note: $\boldsymbol{To\ find\ \ the\ equation\ of\ the\ normal\ to\ the\ curve\ y = f(x),}$
$\boldsymbol{at\ the\ point\ (x_1, y_1)\ use\ the\ formula\ \ y - y_1 = m(x - x_1)\ , where\ \ m = \frac{-1}{f^{/}(x)}}$

Find the equation of the normal to the curve at the specified point below:

28.1 $\boldsymbol{y = sin(x)ln(x)}$ $\boldsymbol{at\ (\pi, 0)}$ • $y - y_1 = m(x - x_1)$ • $x_1 = \pi, \quad y_1 = 0$ • $y - 0 = m(x - \pi)$ • $m = \frac{-1}{f^{/}\ (\pi)}$ • $f^{/}(x) = [\sin(x)]^{/}\ln(x) + \sin(x)\,[\ln(x)]^{/}$ • $f^{/}(x) = \cos(x)\ln(x) + \frac{\sin(x)}{x}$ • $f^{/}(\pi) = -\ln(\pi), \quad m = \frac{1}{\ln(\pi)}$ • $y - 0 = \frac{1}{\ln(\pi)}(x - \pi)$ • $x + yln(\pi) = \pi\ \ or\ x + 1.145y = 3.142$	**28.4** $\boldsymbol{y = ln(cos^2 x)}$ $\boldsymbol{when\ x = \frac{\pi}{4}}$
28.2 $\boldsymbol{y = x^4 ln(x)}$ $\boldsymbol{when\ x = 1}$	**A28.5** $\boldsymbol{y = e^x cos(\pi x)}$ $\boldsymbol{when\ x = 1}$
28.3 $\boldsymbol{y = 4t^3 - 5,\ x = 2t + 3,\ when\ \ t = 1}$	**28.6** $\boldsymbol{y = ln(2t - 5), x = t^2 - 4\ , when\ t = 3}$

M28. Finding the Equation of the Normal to the Curve

Answers

Note: $\boldsymbol{To\ find\ the\ equation\ of\ the\ normal\ to\ the\ curve\ y = f(x),}$
$\boldsymbol{at\ the\ point\ (x_1, y_1)\ use\ the\ formula\ \ y - y_1 = m(x - x_1)\ , where\ \ m = \frac{-1}{f'(x)}}$

Find the equation of the normal to the curve at the specified point below:

28.1 $y = sin(x)ln(x)$ at $(\pi, 0)$

- $y - y_1 = m(x - x_1)$
- $x_1 = \pi, \quad y_1 = 0$
- $y - 0 = m(x - \pi)$
- $m = \frac{-1}{f'(\pi)}$
- $f'(x) = [\sin(x)]' \ln(x) + \sin(x)[\ln(x)]'$
- $f'(x) = \cos(x)\ln(x) + \frac{\sin(x)}{x}$
- $f'(\pi) = -\ln(\pi), \quad m = \frac{1}{\ln(\pi)}$
- $y - 0 = \frac{1}{\ln(\pi)}(x - \pi)$
- $x + yln(\pi) = \pi \ \ or\ x + 1.145y = 3.142$

28.4 $y = ln(cos^2 x)$ when $x = \frac{\pi}{4}$

- $y - y_1 = m(x - x_1)$
- $x_1 = \frac{\pi}{4}, \quad y_1 = ln\left(cos^2\frac{\pi}{4}\right) = -ln2$
- $y + ln2 = m\left(x - \frac{\pi}{4}\right), \quad m = -1 \div f'\left(\frac{\pi}{4}\right)$
- $f'(x) = \frac{-2\cos(x)\sin(x)}{cos^2 x} = -2\tan(x)$
- $f'\left(\frac{\pi}{4}\right) = -2\tan\left(\frac{\pi}{4}\right) = -2, m = 0.5$
- $y + ln2 = 0.5\left(x - \frac{\pi}{4}\right)$
- $2x - 4y = 4ln2 - 0.5\pi\ or$
- $2x - 4y = 1.202\ \ (3dp)$

28.2 $y = x^4 ln(x)$ when $x = 1$

- $y - y_1 = m(x - x_1),$
- $x_1 = 1, \quad y_1 = 1^4 ln(1) = 0$
- $y - 0 = m(x - 1)$
- $m = \frac{-1}{f'(1)}$
- $f'(x) = [x^4]' \ln(x) + x^4[\ln(x)]'$
- $f'(x) = 4x^3\ln(x) + \frac{x^4}{x} = 4x^3\ln(x) + x^3$
- $f'(1) = 4 \times 1^3 \ln(1) + 1^3 = 1, \quad m = -1$
- $y - 0 = -1(x - 1)$
- $x + y = 1$

A28.5 $y = e^x cos(\pi x)$ when $x = 1$

- $y - y_1 = m(x - x_1),$
- $x_1 = 1, \quad y_1 = e^1 cos(\pi) = -e = -2.718$
- $y + e = m(x - 1)$
- $m = -1 \div f'(1)$
- $f'(x) = [e^x]'\cos(\pi x) + e^x[\cos(\pi x)]'$
- $f'(x) = e^x\cos(\pi x) - \pi e^x sin(\pi x)$
- $f'(1) = e^1\cos(\pi) - \pi e^1 sin(\pi) = -e$
- $m = -1 \div f' = 1 \div e = 0.368$
- $y + 2.718 = 0.368(x - 1)$
- $0.368x - y = 3.086$

28.3 $y = 4t^3 - 5, \ \ x = 2t + 3,\ when\ \ t = 1$

- $-y - y_1 = m(x - x_1)$
- $x_1 = 2 \times 1 + 3 = 5, \ y_1 = 4 \times 1^3 - 5 = -1$
- $m = -1 \div f'(5), or\ when\ t = 1$
- $f'(x) = \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$
- $\frac{dy}{dt} = \frac{d}{dt}(4t^3 - 5) = 12t^2$
- $\frac{dx}{dt} = \frac{d}{dt}(2t + 3) = 2$
- $f'(x) = \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = 12t^2 \div 2 = 6t^2$
- $f'(t = 1) = 6 \times 1^2 = 6, m = -1/6$
- $y + 1 = -\frac{1}{6}(x - 5), \quad x + 6y = -1$

28.6 $y = ln(2t - 5), x = t^2 - 4\ , when\ t = 3$

- $-y - y_1 = m(x - x_1)$
- $x_1 = 3^2 - 4 = 5, \ y_1 = ln(2 \times 3 - 5) = 0$
- $m = -1 \div f'(x = 5), or\ when\ t = 3$
- $f'(x) = \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$
- $\frac{dy}{dt} = \frac{d}{dt}[\ln(2t - 5)] = \frac{2}{2t-5}$
- $\frac{dx}{dt} = \frac{d}{dt}(t^2 - 4) = 2t$
- $f'(x) = \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{1}{t(2t-5)}$
- $f'(t = 3) = \frac{1}{3(2\times3-5)} = \frac{1}{3}\ , \quad m = -3$
- $y - 0 = -3(x - 5)\ \ or\ \ 3x + y = 15$

M/E35. Optimisation. Worksheet 3

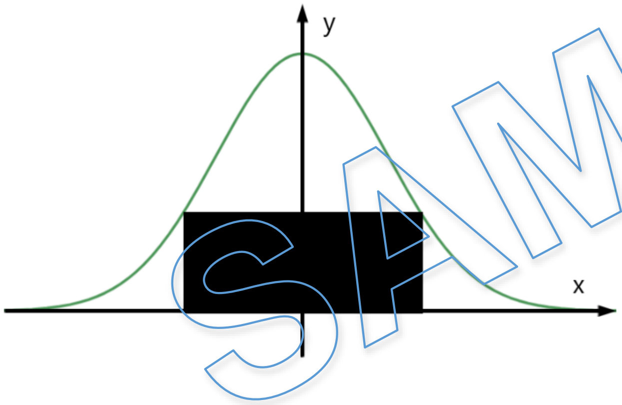
Note: $\boldsymbol{If\ y = f(x),\ at\ the\ stationary\ point\ f^{/}(x) = 0.}$

$$\boldsymbol{At\ the\ local\ maximum\ f^{//}(x) < 0, at\ the\ local\ minimum\ f^{//}(x) > 0}$$

35.1 A rectangle has two vertices at x-axis and other two vertices on the curve

$$y = 3e^{-0.5x^2}$$

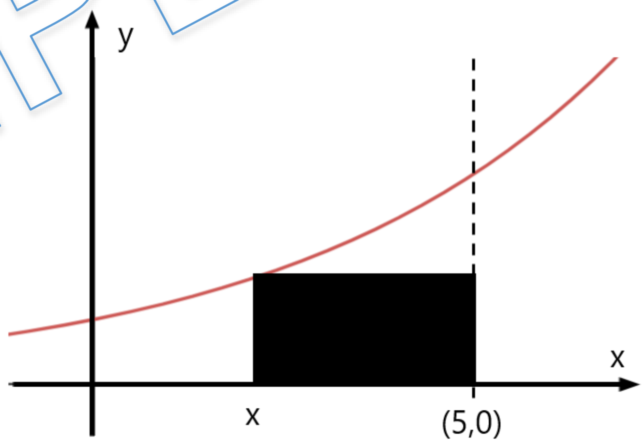
as shown on the graph below. Find the maximum possible area of the rectangle.



35.2 A rectangle has one vertex at (5,0) and the opposite vertex on the curve

$$y = e^{0.2x}$$

where $x \leq 5$, as shown on the graph below. Find the maximum possible area of the rectangle.



M/E35. Optimisation. Worksheet 3. Answers

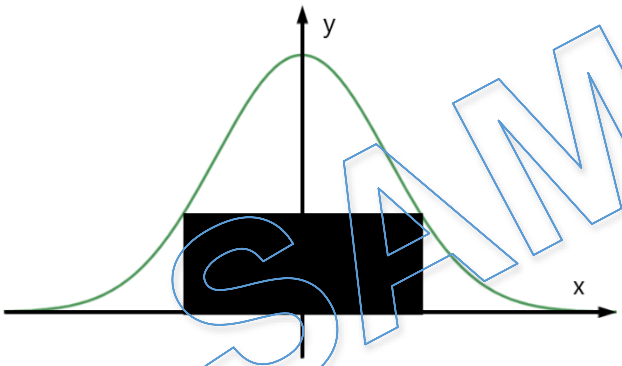
Note: $\boldsymbol{If\ y = f(x),\ at\ the\ stationary\ point\ f^{/}(x) = 0.}$

$\boldsymbol{At\ the\ local\ maximum\ f^{//}(x) < 0, at\ the\ local\ minimum\ f^{//}(x) > 0}$

35.1 A rectangle has two vertices at x-axis and other two vertices on the curve

$$y = 3e^{-0.5x^2}$$

as shown on the graph below. Find the maximum possible area of the rectangle.



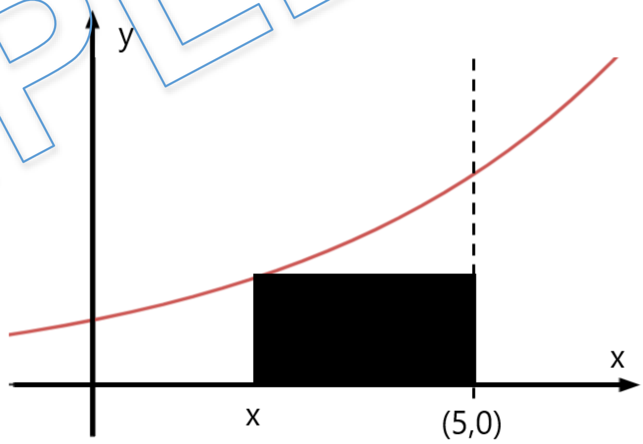
Solution

- $Area = 2x \times 3e^{-0.5x^2} =$
- $6xe^{-0.5x^2}, where\ x > 0$
- $\frac{d}{dx}\left[6xe^{-0.5x^2}\right] =$
  $= 6e^{-0.5x^2} - 6x \times xe^{-0.5x^2}$
- $6e^{-0.5x^2} - 6x \times xe^{-0.5x^2} = 0$
- $6e^{-0.5x^2}[1 - x^2] = 0$
- $x = \pm 1, \rightarrow x = 1$
- $Area\ max = 6xe^{-0.5x^2} = 6e^{-0.5} =$
  $= 3.639\ (3dp)$

35.2 A rectangle has one vertex at (5,0) and the opposite vertex on the curve

$$y = e^{0.2x}$$

where $x \leq 5$, as shown on the graph below. Find the maximum possible area of the rectangle.



Solution

- $Area = (5 - x) \times e^{0.2x}, where$
  $0 \leq x \leq 5$
- $\frac{d}{dx}\left[(5 - x)e^{0.2x}\right] =$
  $-e^{0.2x} + 0.2(5 - x)e^{0.2x} = 0$
- $e^{0.2x}[-1 + 0.2(5 - x)] = 0$
- $x = 0$
- $Area\ max = 5$